Network Analysis & Synthesis (NEC-301)

Unit-I

Signal analysis, complex frequency, network analysis, network synthesis, General characteristics and descriptions of signals, step function and associated wave forms, the unit impulse.

Introduction to network analysis, network elements, initial and final conditions, step and impulse response, solution of network equations.

1.1 Network: An electrical network is an interconnection of electrical elements such as resistors, inductors, capacitors, voltage sources, current sources and switches.

1.2 Network Analysis: The analysis of any network means to obtain response (like voltage across any branch or current passes through any branch) of a network for a known excitation (i.e. input signal).

1.3 Signals: A signal is defined as a function of one or more variables which contain some information, e.g. voltage and current signals are the most common electrical signals. Electrical signals can be expressed as a function of either time or frequency. When these signals are expressed with respect to time, then it is commonly called time domain representation of the signal, on the other hand when they are expressed with respect to frequency, then it is called frequency domain representation of the signal.

1.4 Classifications of Signals:

(a) Periodic and non-periodic signals: A signal that repeats its value at a regular interval of time (called time period) is called periodic signal. It may be defined mathematically as: \( x(t) = x(t+T_0) \), where \( T_0 \) is the time period of the signal.

If a signal does satisfy the above condition, i.e. not repeat itself after a fixed interval of time, it is called non-periodic signal.

(b) Continuous-time and discrete-time signals: Continuous-time signals (or analog signals) may be defined for every value of time for a continuous interval of time. Examples of continuous signals are sine wave, cosine wave, triangular wave etc. Discrete time functions are the sampled version of continuous time functions, in such functions the independent variable is in discrete form. e.g. \( x(t) = 3t \), where \( n = 0,1,2,......... \)

(c) Even and odd functions: A signal is said to be even or symmetrical signal if inversion of time axis doesn’t change its amplitude. These functions are symmetrical about vertical axis in their domain of representation and satisfies the following condition: \( x(-t) = x(t) \).

A signal is said to be odd if it is negative of its reflection and satisfies the relation: \( x(-t) = -x(t) \). In such signals the inversion of time axis inverts amplitude of the signal.

NOTE: An odd signal has amplitude equal to zero at \( t = 0 \).

(d) Deterministic and random signals: Deterministic are those signals which defined completely at specified function of time, there is no uncertainty about its value at any instant of time.

A random signal contains uncertain information about their values. e.g. noise signals.
1.5 Standard test signals (Singular signals):

(a) Step signal: It is defined as:-
\[ f(t) = \begin{cases} 0 & ; t < 0 \\ k & ; t \geq 0 \end{cases} \]
* \( k \)→gain, when its value is one, the step signal is known as unit step signal.

\[ u(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t \geq 0 \end{cases} \]

(b) Ramp signal: The ramp signal is defined as:-
\[ f(t) = \begin{cases} 0 & ; t < 0 \\ mt & ; t \geq 0 \end{cases} \]
* \( m \)→slope of the ramp signal, when its value is one, the step signal is known as unit ramp signal.

\[ r(t) = \begin{cases} 0 & ; t < 0 \\ t & ; t \geq 0 \end{cases} \]

(c) Impulse signal: The impulse signal is defined as:-
\[ f(t) = \begin{cases} 0 & ; t \neq 0 \\ A & ; t = 0 \end{cases} \]
* \( A \)→is the area of the impulse signal and some time called the strength of the impulse.

A unit impulse function is defined for \( A = 1 \) and represents as:-
\[ \delta(t) = \begin{cases} 0 & ; t \neq 0 \\ 1 & ; t = 0 \end{cases} \]

The area of the unit impulse signal is defined as:
\[ \text{Area} = \int_{-\infty}^{\infty} \delta(t) \, dt = 1. \]

In practical case, when a large amplitude (let \( a \)) occurs for a very short duration (let \( \frac{1}{a} \)) then the area of the rectangular pulse is unity, that’s called impulse function.

(d) Gate signal: A rectangular pulse having unit amplitude which is defined for a particular interval is called gate signal and it is defined as:-
\[ G(t) = u(t - a) - u(t - b); \quad a \leq t \leq b \]
\[ G(t) = u(t) - u(t - b); \quad 0 \leq t \leq a \]

(e) Delayed signal: If a signal not starts from \( t = 0 \); then this function (signal) is known as delayed or shifted signal. Some examples of such type of signals is as shown in below:

\[ u(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t \geq a \end{cases} \]
\[ mr(t) = \begin{cases} 0 & ; t < a \\ mt & ; t \geq a \end{cases} \]
\[ \delta(t) = \begin{cases} 0 & ; t \neq a \\ 1 & ; t = a \end{cases} \]

Other Basic signals:

1.6 Waveform Synthesis: There are three possibilities to additions or subtractions of a singular function to be an existing function:

Step ± Step Signals
\( \text{The result of addition or subtraction of a step to step signal is a step signal.} \)
\( \text{The magnitude of the resultant signal is the addition or subtraction respectively of the two steps.} \)

Step ± Ramp signals
\( \text{The resultant of the addition or subtraction of step and ramp signal is a ramp function, shifted by an amount equal to the step.} \)

Ramp ± Ramp Signals
\( \text{The result of addition or subtraction of a ramp to ramp signal is a ramp signal.} \)
\( \text{The slope of the resultant is the algebraic addition of the two slopes.} \)
\( \text{The change in slope occurs at the instant of addition or subtraction.} \)
1.7 Classification of Network Elements:
(a) Active and Passive elements: The elements which are capable to generate electrical energy or have the capability of enhancing the energy level of a signal passing through it are called active elements. e.g. voltage and current sources, op. amps, transistors etc.
On the other hand passive are those elements which consume energy. They having tendency to change the form of applied energy into other form of energy.

(b) Unilateral and Bilateral: Unilateral are those elements in which the direction of current passes through them is changed, then the characteristics or properties of the circuit may also change. e.g. diode, transistors etc.
While the elements which don’t show any change in their response the direction of excitation is changed, are called bilateral elements, e.g. resistors, inductor, capacitor etc.

1.8 Transient behavior of a network: When any changes occurs in any circuit, the circuit behavior changes in terms of their responses with respect to their excitation. These conditions changes may be classified into two categories:
1.8.1 Transient Condition: When changes occurs in any network, for a very short duration of time the circuit response changes rapidly. It may contain peaks of very high amplitudes; this period of time is known as transient period.
1.8.2 Steady State Condition: After transient period, the circuit responses reach its stable value and such condition is called Steady state condition.

1.9 Solution of Circuits Using Differential Equations: In a network generally excitation and response both are the functions of time. We can solve any circuit with the help of differential equations.

1.9.1 First Order Homogeneous Differential Equation:
A first order homogeneous differential equation is given as: \[ \frac{dy(t)}{dt} + Ay(t) = 0 \]
where \( P \rightarrow \text{constant} \)
\[ \Rightarrow \frac{dy(t)}{dt} = -Ay(t) \]
or \[ \frac{dy(t)}{y(t)} = -At + K' \]
on integrating:-
for \( K' = \ln K \)
\[ \ln y(t) = -At + \ln K \]
or \[ \ln y(t) = \ln (Ke^{-At}) \]
So the solution of the equation:-
\[ y(t) = Ke^{-At} \] ..........(1)

1.9.2 First Order Non-Homogeneous Differential Equation:
\[ \frac{dy(t)}{dt} + Ay(t) = Q \] ..........(1) ;
where \( Q \rightarrow \text{may or may not be the function of time} \)
To find the solution of the equation multiply both sides of equation by a factor \( e^{At} \), called integrating factor.
\[ e^{At} \frac{dy(t)}{dt} + A.e^{At}y(t) = Q.e^{At} \]
We know that:
\[ \frac{d}{dx}(a.b) = b \frac{d}{dx}(a) + a \frac{d}{dx}(b) \]
So above equation may be written as:
\[ \frac{d}{dt}(y(t).e^{At}) = Q.e^{At} \]
On integrating:-
\[ y(t).e^{At} = \int Q.e^{At} dt + K \]
So solution of the equation :-
\[ y(t) = e^{-At} \int Q.e^{At} dt + K.e^{-At} \] ..........(2)
The first in above equation is called particular integral, while the second complementary integral.
If \( Q \rightarrow \text{constant} \)
\[ y(t) = e^{At} \int Q \frac{e^{At}}{A} + Ke^{At} \]
\[ y(t) = \frac{Q}{A} + Ke^{At} \] ..........(3)

1.10 Transient Behavior of Any Network:
1.10.1 Initial Conditions in Circuits:
The behavior of any electrical circuit can be examined using differential equations. These equations are given as values of voltage, current, charge or derivatives of these quantities at the instant when network condition changed.
When any signal applied to the network their conditions changed, e.g. a switch act as a step signal and we always assumes that a switch operate in zero time but in such situation circuit conditions doesn’t changes instantly. The network conditions at this instant are called the initial conditions.
**1.10.2 Initial Conditions in Elements:**

**1.10.2 (a) Resistor**

The relation between applied voltage, resulting current and resistance of a conductor is given as: \( V = IR \), this equation is a linear time independent equation, which shows that the current through resistor changes instantaneously if applied voltage changes instantaneously.

**1.10.2 (b) Inductor**

The relation between voltage across any inductor and current passes through it is given as:

\[
\frac{dt}{di} = \frac{L}{v}
\]

For d.c current: \( \frac{dt}{di} = 0 \Rightarrow v_L = 0 \).

So in steady state condition for d.c supply, inductor acts as a short circuit.

Above equation may be represent as:-

\[
i_L = \frac{1}{L} \int v_L dt
\]

Where the limits of the integration are from \( t = -\infty \) to \( t = 0^- \)

so:

\[
i_L = \frac{1}{L} \int_{-\infty}^{t} v_L dt
\]

\[
\Rightarrow i_L = \frac{1}{L} \int_{-\infty}^{t} v_L dt + \frac{1}{L} \int_{0^-}^{t} v_L dt
\]

In above equation the first part of the equation represents its initial value before which the switch is not in operating condition, which is known as initial condition and represents as \( i(0^-) \). So it may be rewrite as:

\[
i_L = i_L(0^-) + \frac{1}{L} \int_{0^-}^{t} v_L dt
\]

At \( t = 0^+ \)

\[
\Rightarrow i_L(0^+) = i_L(0^-) + \frac{1}{L} \int_{0^-}^{t} v_L dt
\]

Initially we assumed that switch acts in zero time, so the second term of the above equation will be zero.

\[
\Rightarrow i_L(0^+) = i_L(0^-)
\]

Hence the current through the inductor cannot be change instantaneously; it takes some time to change its value.

**1.10.2 (c) Capacitor**

The relationship between current through capacitor and voltage across it is given by:

\[
\frac{dv}{dt} = \frac{1}{C} \int i \, dt
\]

If d.c voltage is applied to capacitor, \( \frac{dv}{dt} \) becomes zero, so capacitor current becomes zero. In steady state for d.c supply, capacitor acts as an open circuit.

Above equation may be rewrite as:

\[
v_c = \frac{1}{C} \int_{-\infty}^{t} i \, dt
\]

Above equation may be write as:

\[
v_c = \frac{1}{C} \int_{-\infty}^{t} i \, dt + \frac{1}{C} \int_{0^-}^{t} i \, dt
\]

In above equation the first part of the equation represents its initial value before which the switch is not in operating condition, which is known as initial condition and represents as \( v_c(0^-) \). So it may be rewrite as:

\[
v_c = v_c(0^-) + \frac{1}{C} \int_{0^-}^{t} i \, dt
\]

At \( t = 0^+ \)

\[
v_c(0^+) = v_c(0^-) + \frac{1}{C} \int_{0^-}^{t} i \, dt
\]

Initially we assumed that switch acts in zero time, so the second term of the above equation will be zero.

\[
\Rightarrow v_c(0^+) = v_c(0^-)
\]

Hence voltage across capacitor cannot be change instantaneously; it takes some time to change its value.

**1.11 DC Response of Series R-L Circuit:**

An R-L circuit is shown in figure. At \( t = 0^- \), switch is about to be close but not fully operating condition. Since initial current passes through inductor is zero, so for such condition initial value of current i.e. \( i(0^-) = 0 \).

At \( t = 0^- \):

\[
V = i.R + L \frac{di}{dt}
\]

\[
\Rightarrow \frac{V}{R} = i + L \frac{di}{R \, dt}
\]
\[ \frac{V}{R} - i = \frac{L}{R} \frac{di}{dt} \]

Above equation may be rearrange as:

\[ \frac{R}{L} \frac{di}{dt} = \frac{V}{R} - i \]

On integrating:

\[ \frac{R}{L} t = -\ln \left( \frac{V}{R} - i \right) + K \]

To find the value of “K”:

at \( t = 0 \) \( i = I_0 = 0 \)

So above equation becomes:

\[ \frac{R}{L} (0) = -\ln \left( \frac{V}{R} - 0 \right) + K \quad \Rightarrow \quad K = \ln \left( \frac{V}{R} \right) \]

So

\[ \frac{R}{L} t = -\ln \left( \frac{V}{R} - i \right) + \ln \left( \frac{V}{R} \right) = -\ln \left( \frac{V}{R} - i \right) \]

On taking antilog of the equation we get:

\[ e^{-\frac{R}{L} t} = \left( \frac{V}{R} \right) \frac{V}{R - i} \]

\[ \Rightarrow \quad \frac{V}{R} - i = \frac{V}{R} e^{-\frac{R}{L} t} \]

Hence the expression for \( i \) becomes:

\[ i = \frac{V}{R} \left( 1 - e^{-\frac{R}{L} t} \right) \]  

This expression of current contains two parts, first part of the equation i.e. \( \frac{V}{R} \) represents steady state part of current and second part of the equation represents its transient behavior.

The response of the current may be representing as shown in figure.

At the time instant where \( t = \frac{L}{R} \), current reaches a value of 63.2% of its final value; this value of time is represented by \( \tau \) and known as time constant of the system.

\[ V \]

1.12 Current Decay in Series R-L Circuit:

Consider at \( t = 0^+ \), switch is at position ‘a’ for very long time thus network reaches its steady state.

So in such condition initial current through inductor will be:

\[ i_L(0^-) = I_0 = \frac{V}{R} = i(0^+) \]

\[ v_L = \frac{L}{\frac{d}{dt} \left( 1 - e^{-\frac{R}{L} t} \right)} \]

Now at \( t = 0 \), switch position changed to ‘b’. In such condition; from KVL:

\[ L \frac{di}{dt} + iR = 0 \quad \Rightarrow \quad L \frac{di}{dt} = -iR \]
1.13 Response of Series R-C Circuit:

A series R-C circuit shown in figure. Let the capacitor is initially discharged so in such condition voltage across capacitor is zero. At \( t = 0^+ \) switch is about to be close but not fully operating condition. Since initial current passes through inductor is zero, so for such condition initial value of current \( i.e. v_c(0^+) = 0 \).

After switching instant, from KVL:
\[
V = V_R + v_C = iR + v_C
\]

where \( i = C \frac{dv_C}{dt} \)
\[
\Rightarrow V = RC \frac{dv_C}{dt} + v_C
\]
\[
\Rightarrow \frac{dv_C}{dt} = \frac{1}{RC} \frac{V - v_C}{V - v_C}
\]

Integrating both sides of the above equation, we get:
\[
\int \frac{dv_C}{V - v_C} = \ln \left( \frac{V - v_C}{V} \right) = \frac{t}{RC}
\]

Now at \( t = 0 \); \( v_C = 0 \) \( \Rightarrow K = \ln V \).

So the equation becomes:
\[
\frac{t}{RC} = \ln \left( \frac{V}{V - v_C} \right)
\]

Taking antilog of the above equation.
\[
e^{t/RC} = \frac{V}{V - v_C}
\]

On solving the above equation; we get the expression for voltage across capacitor as:
\[
v_C = V \left[ 1 - e^{-t/RC} \right]
\]  

Above is the solution across capacitor, first term in the equation gives steady state value of voltage across capacitor and second term gives transient portion of voltage across capacitor.

Consider the steady state is achieved, total charge on capacitor is \( Q \) coulombs.

\[
V = \frac{Q}{C}\]

For \( q \rightarrow \text{instantaneous charge} \), then the value of instantaneous voltage across the capacitor:- \( v_c = \frac{q}{C} \).
So from eq. (13) :\[
\frac{q}{C} = \frac{Q}{C} \left[ 1 - e^{-\frac{t}{RC}} \right]
\]
\[\Rightarrow \quad q = Q \left[ 1 - e^{-\frac{t}{RC}} \right] \quad \ldots \ldots (14)
\]
Above eq. (14) shows that charge on the capacitor having similar behavior as voltage across it.

Now the expression for current:-
\[i = \frac{V}{R} e^{-\frac{t}{RC}} \quad \ldots \ldots (15)
\]

The basic behavior of voltage and charging current is as shown in figure. It is clear from the above equation and figure that voltage in a $RC$ network increases exponentially.

In figure the instant at \(t = RC\); the voltage across capacitor becomes 63.2% of the applied voltage, this instant is called time constant of the network and represented as $\tau$.

### 1.14 Discharge of Capacitor:
Consider in shown figure, switch $K$ is moved from position 1 to 2.
Assume that before $t = 0$, the capacitor was fully charged to voltage $V$ and for $t > 0$ it will discharge through the resistor $R$.
For $t > 0$; from K.V.L-
\[0 = V_R + v_c \]
where:- $V_R$: voltage across resistor $v_c$: voltage across capacitor.
The above equation may be rewrite as:-
\[v_c = -V_R = -iR \]
\[\therefore \quad v_c = -RC \frac{dv_c}{dt} \quad (\because \quad i = C \frac{dv_c}{dt})
\]
\[\Rightarrow \quad \frac{dt}{RC} = \frac{dv_c}{v_c} \quad \ldots \ldots (16)
\]
Integrating both sides of the equation:
\[\frac{t}{RC} = -\ln v_c + K \quad \ldots \ldots (16)
\]
At $t = 0$, $v_c = V$ (initial condition) so the above equation becomes:
\[0 = -\ln V + K \quad \Rightarrow \quad K = \ln V
\]
Hence the above eq. (16) may be rewrite as:
\[\frac{t}{RC} = \frac{V}{v_c}
\]
\[\therefore \quad v_c = e\frac{t}{RC} \quad \ldots \ldots (17)
\]
In such case there is no source in the circuit, so the discharging current $i$ is given by:
\[i = \frac{V}{R} \Rightarrow \frac{v_c}{R} e^{\frac{t}{RC}} \quad \ldots \ldots (18)
\]
Then variation of this current with respect to time is shown in figure.

#### 1.15 Step Response of Series R-L-C Circuit:
In series $R-L-C$ circuit, there are two energy storing elements which represent a second order differential equation.
A series $R-L-C$ circuit is shown in fig. The switch is closed at $t = 0$, in such case a step voltage of magnitude $V$ gets applied to the circuit.
Applying KVL:
\[L \frac{di}{dt} + Ri + \frac{1}{C} \int idt = V
\]
Differentiating both sides of the above equation:
\[L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = 0
\]
\[\Rightarrow \quad \frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0
\]
\[ s^2 + \frac{R}{L}s + \frac{1}{LC}i = 0 \quad \text{........(19)} \]

Above eq. (19) is called **Characteristic or auxiliary equation**, and the response of the circuit depends on the nature of the roots of this equation.

The general solution of this equation is:

\[ i(t) = K_1e^{s_1t} + K_2e^{s_2t} \]

or

\[ i(t) = K_1e^{-(\alpha+j\omega)t} + K_2e^{-(\alpha-j\omega)t} \quad \text{........(20)} \]

Where \(s_1\) and \(s_2\) are the roots of the characteristics equation.

1.16 Complex Frequency:

The complex frequency is denoted by variable \(s\) and it is denoted as:

\[ s = \sigma + j\omega \quad \text{........(21)} \]

Where \(\sigma\) \(\rightarrow\) real part and called **attenuation** or **growth constant**. It denotes the increment or decrement in amount of amplitude of the signal.

\(\omega\) \(\rightarrow\) angular frequency.

Let a signal is given as: \(x(t) = Ae^{\sigma t}\)

From eq. (21)

\[ x(t) = Ae^{(\sigma+j\omega)t} \]

\[ x(t) = Ae^{\sigma t}e^{j\omega t} \]

Above equation can be modified as:

\[ x(t) = Ae^{\sigma t}[\cos(\omega t + j\sin(\omega t)) \]

\[ x(t) = Ae^{\sigma t}\cos(\omega t + jAe^{\sigma t}\sin(\omega t)) \quad \text{........(22)} \]

Eq. (22) shows that a signal can be represented in their real and imaginary parts as:

\[ \text{Re}[x(t)] = Ae^{\sigma t}\cos(\omega t) \]

\[ \text{Im}[x(t)] = Ae^{\sigma t}\sin(\omega t) \]

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**Assignment in Unit-I (Previous Year Questions)**

Q.1 The switch in circuit of figure has been closed for a very long time. It opens at \(t = 0\). Find \(V_c(t)\) for \(t > 0\) using differential equation approach.

Q.2 Consider the network shown in figure. The switch is initially closed for a long time. The switch is opened at \(t = 0\). Find differential equation relating \(i_L(t)\) with \(V(t)\) and also evaluate initial conditions.

Q.3 In the network shown in figure, the switch is in position 1 long enough to establish steady state conditions. At \(t = 0\), the switch is moved to position 2. Find the current in the circuit.

Q.4 Two ramp functions are given by:

\[ F_1(t) = mtu(t) \] and \[ F_2(t) = (t-a)u(t-a) \]

where \(m\) and \(n\) are two slopes (+ve) and \(m > n\). Draw the final waveform from adding these two functions.

Q.5 Find the current when the voltage applied is

\[ v(t) = 2e^{0.5t}u(t) \] and \[ i(0) = 0 \] and \(v_c(0) = 0\).